Multivariable PID control with set-point weighting via BMI optimisation

F.D. Bianchi, and R.S. Sánchez Peña

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Interpolation for gain scheduled control with guarantees ⋆

Fernando D. Bianchi [a,∗] and Ricardo S. Sánchez Peña [b]

[a] *IREC Catalonia Institute for Energy Research Josep Pla, B2, Pl. Baixa, E-08019 Barcelona, Spain.*

[b] *CONICET and Instituto Tecnológico de Buenos Aires (ITBA) Av. E. Madero 399, (C1106ACD) Buenos Aires, Argentina*

Abstract

Here, a methodology is presented which considers the interpolation of LTI controllers designed for different operating points of a nonlinear system in order to produce a gain-scheduled controller. Guarantees of closed-loop quadratic stability and performance at intermediate interpolation points are presented in terms of a set of LMIs. The proposed interpolation scheme can be applied in cases where the system must remain at the operating points most of the time and the transitions from one point to another rarely occur, e.g. chemical processes, satellites.

Key words: Gain scheduling, interpolation, Youla parametrization, linear matrix inequality, linear parameter varying system

1 Introduction

Gain-scheduling has been used successfully to control nonlinear systems for many decades and in many different applications, such as autopilots and chemical processes (Rugh & Shamma, 2000). It consists in selecting a family of operating points or more generally regions, where the system can be described by a linear model. A linear controller is designed for each region which should guarantee performance and robustness in that region. Finally the controllers are changed according to a physical parameter measured in real time, which detects in what region the system is working at each time. The change of controllers can be implemented either gradually by interpolation of certain parameters or by switching.

In practice, the switching among controllers may create instability of the closed loop system (Liberzon, 2003). Unstable modes and degraded performance may come from the transition dynamics, which are not contained in the information provided by each linear model. Usually, a way to mitigate this problem is to impose certain dwell time (Hespanha & Morse, 1999). However, this is not able to prevent the undesirable transients, which may require complex algorithms to reduce their negative effects.

On the other hand, interpolated gain scheduled controllers provide a smooth change among them. In general, this is a fairly simple solution in the cases of SISO problems or fixed structure controllers, such as PIDs or lateral-directional aircraft control, due to the fact that only certain fixed parameters are interpolated, *e.g.* gains, poles, numerator/denominator coefficients. However, in more general cases where the set of controllers have been designed independently or are MIMO models, the implementation of the parameter interpolation is not as simple. In addition, in these cases it is convenient to interpolate the controller state-space realization instead of parameters from its transfer matrix.

Stability and performance guarantees in the whole operating envelope can be obtained using linear parameter varying (LPV) systems theory (Apkarian, Gahinet & Becker, 1995; Wu, Yang, Packard & Becker, 1996). The main problem of this method is the computational effort needed to obtain an LPV controller which limits its use from low to medium order systems. In addition, in many fields, *e.g.* aerospace, there is a strong interest of practitioners in using the gain-scheduling method, based on optimized designs at different operating points.

For controllers designed independently for each point,

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∗ Corresponding author. Phone: +34 93 3562615 / Fax: +34 93 3563802.

Email addresses: fbianchi@irec.cat (Fernando D. Bianchi), rsanchez@itba.edu.ar (Ricardo S. Sánchez Peña).

previous results have focused on stability (Stilwell & Rugh, 2000; Chang & Rasmussen, 2008) or on controller switching instead (Hespanha & Morse, 2002; Blanchini, Miani & Mesquine, 2009). In particular in Chang & Rasmussen (2008), the Youla parametrization has been used, but a network of controllers is produced which significantly increases the order of the resulting gain-scheduled control. Some recent results consider the performance problems by establishing an adequate controller initial condition when switching (Hespanha, Santesco & Stewart, 2007) or by injecting stabilizing signals among the local controllers, based on bumpless and antiwindup transfer compensators (Hencey & Alleyne, 2009). There are no results focused on both, stability and performance, based on the adequate selection of the state-space realizations for interpolation.

This paper focuses on formulating a stability preserving interpolation scheme with a performance level guarantee in the state-space framework. The aim is to obtain gain scheduled controllers with similar stability properties as LPV versions and with the possibility of tuning each LTI controller independently. Next section presents the problem statement and Section 3 the main results, illustrated by a short example in Section 4. The paper ends in Section 5 with some concluding remarks.

2 Problem statement

Consider the set of linear models

$$G_i(s) = \left[\begin{array}{c|cc} A_i & B_{1,i} & B_2 \\ \hline C_{1,i} & D_{11,i} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right], \qquad i \in \mathbb{I}_{n_p} \tag{1}$$

describing the local dynamic behavior of a nonlinear or time-varying system at each operating point parameterized by $\rho_i \in \mathcal{P}$, with $A_i \in \mathbb{R}^{n\times n}$ and $\mathbb{I}_{n_p} = \{1, \dots, n_p\}$. The set of points $\{\rho_1, \dots, \rho_{n_p}\}$ divides the region $\mathcal{P}$ in a set of subregions $\mathcal{P}_j$ defined by the vertices $\mathcal{V}_j \subseteq \{\rho_1, \dots, \rho_{n_p}\}$ as illustrated in Figure 1. Then, any point $\rho \in \mathcal{P}_j$ can be expressed as a convex combination of the vertices $\mathcal{V}_j$, *i.e.*,

$$\rho = \sum_{i=1}^{n_p} \alpha_i \rho_i \tag{2}$$

where $\alpha_1 + \cdots + \alpha_{n_p} = 1$ and $\alpha_i \geq 0, \forall \rho_i \in \mathcal{V}_j$, $\alpha_i = 0, \forall \rho_i \notin \mathcal{V}_j$.

The local dynamics at any point $\rho \in \mathcal{P}_j$ is assumed to be described as a linear combination of the state-space realizations corresponding to the vertices $\mathcal{V}_j$

$$G(\rho) : \begin{cases} \dot{x} = A(\rho)x + B_1(\rho)w + B_2 u, \\ z = C_1(\rho)x + D_{11}(\rho)w + D_{12}u, \\ y = C_2 x + D_{21} w, \end{cases} \tag{3}$$

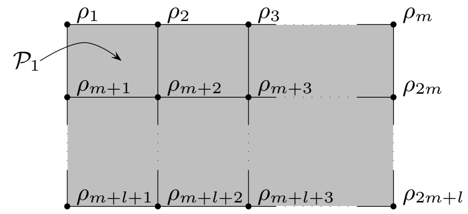

Figure 1. Example of division of the region $\mathcal{P}$

where

$$\begin{bmatrix} A(\rho) & B_1(\rho) \\ C_1(\rho) & D_{11}(\rho) \end{bmatrix} = \sum_{i=1}^{n_p} \alpha_i(\rho) \begin{bmatrix} A_i & B_{1,i} \\ C_{1,i} & D_{11,i} \end{bmatrix}$$

and $\alpha_i(\rho)$ is the coordinate corresponding to ρ_i.

According to (2), only the matrices corresponding to $\rho_i \in \mathcal{V}_j$ are needed to compute system (3). This class of models is called *piecewise affine* LPV systems (Lim & How, 2003), which includes the classical affine LPV models. The assumption that B_2, C_2, D_{12} and D_{21} are constant does not impose a serious constraints. This can be fulfilled by simply filtering the input u and/or the output y (see Apkarian et al., 1995).

It is assumed that there exists a stabilizing linear controller designed beforehand and independently for each plant $G_i(s)$

$$K_i(s) = \left[\begin{array}{c|c} A_{k,i} & B_{k,i} \\ \hline C_{k,i} & D_{k,i} \end{array}\right], \qquad i = 1, \dots, n_p \tag{4}$$

which achieves certain performance specifications, with $A_{k,i} \in \mathbb{R}^{n_c \times n_c}$. This differs from other synthesis procedures applicable to the plant (3) like the gridding method proposed by (Wu et al., 1996) or the switching LPV framework by (Lim & How, 2003), where the local controllers are computed simultaneously.

Then, the objective is to formulate an interpolation scheme for the state-space realizations (4) such that the gain-scheduled controller

$$K(\rho) : \begin{cases} \dot{x}_k = A_k(\rho)x_k + B_k(\rho)y, \\ u = C_k(\rho)x_k + D_k(\rho)y, \end{cases} \tag{5}$$

stabilizes the plant $G(\rho)$ defined in (3) at any point $\rho \in \mathcal{P}$, with $A_k(\rho) \in \mathbb{R}^{n_k \times n_k}$. Note that the order of the local controllers (4) may differ from the order of the gain-scheduled controller (5) (*i.e.*, in general, $n_c \neq n_k$).

3 Main results

The following lemma provides a systematic method to find a quadratically stable interpolation of several Hurwitz matrices. If the set of matrices A_i represents the local dynamics of an LPV system at the vertices of a convex hull $\mathrm{co}\{\rho_1, \ldots, \rho_{n_p}\}$, the following results states that given a set of Hurwitz matrices, it is always possible to construct a quadratically stable affine LPV matrix.

Lemma 3.1 *Given a set of matrices A_i associated to each vertex of the convex hull $\Theta = \mathrm{co}\{\rho_1, \ldots, \rho_{n_p}\}$, the following statements are equivalent*

i) *A_i is Hurwitz for all $i \in \mathbb{I}_{n_p}$,*

ii) *there exist n_p matrix transformations T_i such that the LPV matrix*

$$\tilde{A}(\rho) = \sum_{i=1}^{n_p} \alpha_i(\rho)\tilde{A}_i = \sum_{i=1}^{n_p} \alpha_i(\rho) T_i A_i T_i^{-1} \qquad (6)$$

is quadratically stable for all $\rho \in \Theta$, with $\alpha_i(\rho) = \alpha_i$ in $\rho = \sum_{i=1}^{n_p} \alpha_i \rho_i$ such that $\sum_{i=1}^{n_p} \alpha_i = 1$.

Proof: i) $\Rightarrow$ ii), if A_i is Hurwitz, then $\exists\ X_i > 0$ such that $X_i A_i + A_i^T X_i < 0$, $i \in \mathbb{I}_{n_p}$. According to (Hespanha & Morse, 2002), it is always possible to find state transformations T_i (*e.g.* $T_i = X_i^{1/2}$) such that

$$X\tilde{A}_i + \tilde{A}_i^T X < 0, \quad \forall\, i \in \mathbb{I}_{n_p} \qquad (7)$$

for a common $X > 0$, with $\tilde{A}_i = T_i A_i T_i^{-1}$. Finding the coordinates $\alpha_i(\rho)$'s, ρ as a convex combination of the vertices of Θ, the LPV matrix (6) can be constructed. Based on $\alpha_i \geq 0$, $\forall\, i \in \mathbb{I}_{n_p}$, inequalities (7) and linearity,

$$X\left(\sum_{i=1}^{n_p} \alpha_i(\rho)\tilde{A}_i\right) + \left(\sum_{i=1}^{n_p} \alpha_i(\rho)\tilde{A}_i\right)^T X < 0 \qquad (8)$$

and thus the quadratical stability of $\tilde{A}(\rho)$ is proved.

ii) $\Rightarrow$ i), take $\rho = \rho_m$, with ρ_m one of the vertex of Θ, then $\alpha_m = 1$, and $\alpha_i = 0$, $\forall\, i \neq m$. Therefore, $\tilde{A}(\rho) = \tilde{A}_m$ and from (8) it can be concluded that $\tilde{A}_m$ is Hurwitz and thus A_m. □

3.1 Quadratically stable interpolation

Based on the previous results and the Youla parametrization, a quadratically stable interpolation procedure is formulated. It computes non minimum state-space realizations of the controller matrices which leads to a quadratic stabilizing gain scheduled controller when they are linearly interpolated. The computation of these state-space realizations is based on an LMI optimization problem. This is an extension of the results in Hespanha & Morse (2002), using a technical tool from Xie & Eisaka (2004).

Theorem 3.2 *Given the set of plants (1) and the set of stabilizing controllers (4), if there exist positive definite matrices $X_1 \in \mathbb{R}^{n\times n}$, $X_{2,i} \in \mathbb{R}^{n_q\times n_q}$ and $X_3 \in \mathbb{R}^{n\times n}$, matrices V_i and W_i such that*

$$(X_1 A_i + W_i C_2) + (X_1 A_i + W_i C_2)^T < 0, \qquad (9)$$

$$(X_{2,i} A_{q,i}) + (X_{2,i} A_{q,i})^T < 0, \qquad (10)$$

$$(A_i X_3 + B_2 V_i) + (A_i X_3 + B_2 V_i)^T < 0 \qquad (11)$$

for all $i \in \mathbb{I}_{n_p}$, with

$$A_{q,i} = \begin{bmatrix} A_i + B_2 D_{k,i} C_2 & B_2 C_{k,i} \\ B_{k,i} C_2 & A_{k,i} \end{bmatrix},$$

then the gain scheduled controller (5) quadratically stabilizes the plant (3) for all $\rho \in \mathcal{P}$, and its state-space matrices are

$$A_k(\rho) = \sum_{i=1}^{n_p} \alpha_i(\rho) \begin{bmatrix} A_i + B_2 F_i + H_i C_2 - B_2 D_{k,i} C_2 & B_2 \tilde{C}_{q,i} \\ -\tilde{B}_{q,i} C_2 & \tilde{A}_{q,i} \end{bmatrix}, \qquad (12)$$

$$B_k(\rho) = \sum_{i=1}^{n_p} \alpha_i(\rho) \begin{bmatrix} B_2 D_{k,i} - H_i \\ \tilde{B}_{q,i} \end{bmatrix}, \qquad (13)$$

$$C_k(\rho) = \sum_{i=1}^{n_p} \alpha_i(\rho) \begin{bmatrix} F_i - D_{k,i} C_2 & \tilde{C}_{q,i} \end{bmatrix}, \qquad (14)$$

$$D_k(\rho) = \sum_{i=1}^{n_p} \alpha_i(\rho) D_{k,i}, \qquad (15)$$

$$\tilde{A}_{q,i} = T_i A_{q,i} T_i^{-1}, \qquad \tilde{B}_{q,i} = T_i \begin{bmatrix} B_2 D_{k,i} - H_i \\ B_{k,i} \end{bmatrix},$$

$$\tilde{C}_{q,i} = \begin{bmatrix} D_{k,i} C_2 - F_i & C_{k,i} \end{bmatrix} T_i^{-1},$$

$T_i = X_{2,i}^{1/2}$, $F_i = V_i X_3^{-1}$, $H_i = X_1^{-1} W_i$, $(i \in \mathbb{I}_{n_p})$.

Proof: According to the Youla parametrization, any stabilizing controller $\tilde{K}_i(s)$ for the plant $G_i(s)$ can be expressed as a linear fractional transformation (LFT)

$$\tilde{K}_i(s) = \mathscr{F}_\ell(J_i(s), Q_i(s)) = \left[\begin{array}{c|c} \tilde{A}_{k,i} & \tilde{B}_{k,i} \\ \hline \tilde{C}_{k,i} & \tilde{D}_{k,i} \end{array}\right],$$

$$J_i(s) = \left[\begin{array}{c|cc} A_i + B_2 F_i + H_i C_2 & -H_i & B_2 \\ \hline F_i & 0 & I \\ -C_2 & I & 0 \end{array}\right], \qquad (16)$$

$$Q_i(s) = \left[\begin{array}{c|c} A_{q,i} & B_{q,i} \\ \hline C_{q,i} & D_{q,i} \end{array}\right],$$

with $A_{q,i}$ a Hurwitz matrix. After straightforward manipulations, it can be proved that if

$$Q_i(s) = \left[\begin{array}{cc|c} A_i + B_2 D_{k,i} C_2 & B_2 C_{k,i} & B_2 D_{k,i} - H_i \\ B_{k,i} C_2 & A_{k,i} & B_{k,i} \\ \hline D_{k,i} C_2 - F_i & C_{k,i} & D_{k,i} \end{array}\right], \tag{17}$$

the controllers $\tilde{K}_i(s)$ are I/O equivalent to the original local controllers $K_i(s)$. Note that $A_{q,i}$ corresponds to the A matrix of the closed loop system $\mathscr{F}_\ell(G_i(s), K_i(s))$, hence $Q_i(s)$ is stable if the controller $K_i(s)$ stabilizes $G_i(s)$. Then, replacing the controller matrices (LFT interconnection between (16) and (17)) in the closed loop matrix

$$A_{c\ell,i} = \begin{bmatrix} A_i + B_2 \tilde{D}_{k,i} C_2 & B_2 \tilde{C}_{k,i} \\ \tilde{B}_{k,i} C_2 & \tilde{A}_{k,i} \end{bmatrix}$$

and applying a similarity transformation, the following result is obtained

$$A_{c\ell}(\rho) = \sum_{i=1}^{n_p} \alpha_i(\rho) \begin{bmatrix} A_{H,i} & 0 & 0 \\ B_{q,i} C_2 & A_{q,i} & 0 \\ B_{H,i} C_2 & B_2 C_{q,i} & A_{F,i} \end{bmatrix}, \tag{18}$$

with $A_{H,i} = A_i + H_i C_2$, $A_{F,i} = A_i + B_2 F_i$ and $B_{H,i} = B_2 D_{q,i} - H_i$. Next, to ensure quadratic stability at any point in $\mathcal{P}$ a matrix $X_{c\ell} > 0$ must be computed such that $X_{c\ell} A_{c\ell}(\rho) + A_{c\ell}(\rho)^T X_{c\ell} < 0$. Due to the block triangular structure of $A_{c\ell}(\rho)$ (Lemma 2 in Xie & Eisaka (2004)), the previous inequality is satisfied if the following three ones hold

$$\sum_{i=1}^{n_p} \alpha_i(\rho)(X_1 A_{H,i} + A_{H,i}^T X_1) < 0, \tag{19}$$

$$\sum_{i=1}^{n_p} \alpha_i(\rho)(Y_2 A_{q,i} + A_{q,i}^T Y_2) < 0, \tag{20}$$

$$\sum_{i=1}^{n_p} \alpha_i(\rho)(Y_3 A_{F,i} + A_{F,i}^T Y_3) < 0, \tag{21}$$

with $X_{c\ell} = \operatorname{diag}(X_1, Y_2, Y_3) \in \mathbb{R}^{(2n+n_q)\times(2n+n_q)}$.

Taking into account that the $A_{q,i}$'s are Hurwitz matrices by construction and the result in Lemma 3.1, if $X_{2,i} = T_i^T Y_2 T_i$, the inequality (20) is equivalent to (10). On the other hand, using the vertex property (see Apkarian et al., 1995), (19) and (21) can be reduced to prove the existence of positive definite matrices X_1 and $X_3 = Y_3^{-1}$ which satisfy (9) and (11) at each $i \in \mathbb{I}_{n_p}$ with $W_i = X_1 H_i$ and $V_i = F_i X_3$. □

Note that $n_q = n + n_c$ and $n_k = n + n_q = 2n + n_c$ and the resulting gain-scheduled controller order is independent of the number of points n_p. This is more efficient than previous results (Chang & Rasmussen (2008)) based on the Youla parametrization which produce a network of controllers with a final order directly proportional to n_p.

3.2 Performance during transitions

In general, all results on gain scheduling center their attention only on preserving the stability during transitions among controllers. However, a stability preserving interpolation does not necessarily guarantee the performance levels achieved at any design point ρ_i. The reason can be found in the fact that it is not simple to obtain a controller providing a uniform performance level when each controller is designed independently. The LPV framework gives a complete solution to this problem. However, all controllers are designed simultaneously which may limit the local performance levels.

Here, the problem is posed as the search for the state-space realizations of $\tilde{K}_i(s)$'s that achieve the best performance possible in the intermediate points without degrading the performance at the design points ρ_i. This constraints depends on the particular criterium employed to measure the performance specifications. In the following paragraphs the $\mathcal{H}_\infty$ performance case is discussed, although other cases can be addressed in a similar way. Imposing a block diagonal structure on $X_{c\ell}$, at the expense of certain conservatism, the search for the realizations reduces to the following result.

Theorem 3.3 *Given the set of plants (1) and the set of controllers (4) such that $\|\mathscr{F}_\ell(G_i(s), K_i(s))\|_\infty < \gamma_i$. If there exist positive definite matrices X_1, $X_{2,i}$ and X_3 and matrices F_i and H_i ($i \in \mathbb{I}_{n_p}$) such that the n_p matrix inequalities (22) are satisfied, then the controller (5) with state-space realization (12)-(15) quadratically stabilizes plant (3), $\forall \rho \in \mathcal{P}$ and guarantees a performance level $\|z\|_2 < \gamma \|w\|_2$, with $\gamma_i \leq \gamma$, $\forall\, i \in \mathbb{I}_{n_p}$.*

Proof: Define $X_{c\ell} = \operatorname{diag}(X_1, Y_2, Y_3) \in \mathbb{R}^{(2n+n_q)\times(2n+n_q)}$ and replace the parameter matrices by

$$\tilde{A}_{q,i} = T_i A_{q,i} T_i^{-1}, \quad \tilde{B}_{q,i} = T_i B_{q,i}, \quad \tilde{C}_{q,i} = C_{q,i} T_i^{-1}.$$

in the closed loop matrices $A_{c\ell,i}$ in (18), and in

$$B_{c\ell}(\rho) = \sum_{i=1}^{n_p} \alpha_i(\rho) \begin{bmatrix} B_{1,i} + H_i D_{21} \\ \tilde{B}_{q,i} D_{21} \\ B_{H,i} D_{21} \end{bmatrix},$$

$$C_{c\ell}(\rho) = \sum_{i=1}^{n_p} \alpha_i(\rho) \begin{bmatrix} C_{1,i} + D_{12} D_{k,i} C_2 & D_{12} \tilde{C}_{q,i} & C_{1,i} + D_{12} F_i \end{bmatrix},$$

$$
\begin{bmatrix}
X_1 A_{H,i} + (\star) & (B_{q,i}C_2)^T X_{2,i} & C_2^T B_{H,i}^T & X_1(B_{1,i} + H_i D_{21}) & (C_1 - D_{12}D_{q,i}C_2)^T \\
\star & X_{2,i}A_{q,i} + (\star) & (B_2 C_{q,i})^T & X_{2,i}B_{q,i}D_{21} & (D_{12}C_{q,i})^T \\
\star & \star & A_{F,i}X_3 + (\star) & B_{H,i}D_{21} & X_3(C_{1,i} + D_{12}F_i)^T \\
\star & \star & \star & -\gamma I & (D_{11} + D_{12}D_{q,i}D_{21})^T \\
\star & \star & \star & \star & -\gamma I
\end{bmatrix} < 0 \qquad (22)
$$

$$
D_{c\ell}(\rho) = \sum_{i=1}^{n_p} \alpha_i(\rho)(D_{11,i} + D_{12}D_{k,i}D_{21}).
$$

Next, apply the congruence transformation $P = \mathrm{diag}(I, T_i, I, I, I)$ in the BRL inequality

$$
\begin{bmatrix}
X_{c\ell}A_{c\ell} + A_{c\ell}^T X_{c\ell} & X_{c\ell}B_{c\ell} & C_{c\ell}^T \\
B_{c\ell}^T X_{c\ell} & -\gamma I & D_{c\ell}^T \\
C_{c\ell} & D_{c\ell} & -\gamma I
\end{bmatrix} < 0, \qquad (23)
$$

With the previous closed-loop matrices and defining $X_{2,i} = T_i^T Y_2 T_i > 0$ and $X_3 = Y_3^{-1}$, equation (23) becomes the matrix inequality (22) where $\star$ represents the matrix symmetric elements. □

Note that matrices $B_{q,i}$ and $C_{q,i}$ depend on the gains H_i and F_i, respectively, and both are also affected by the transformation T_i. Therefore, this approach produces a non-convex problem when finding these variables simultaneously. Nevertheless, note that the I/O behavior at all vertices is unaffected by the particular selection of H_i and F_i, based on the Youla parametrization results. Therefore, it is sensible to replace the matrices obtained from the stabilization problem in Section 3.1. As a consequence the problem can be transformed into two convex ones.

(1) Given the controllers $K_i(s)$, find X_1, X_3 and the n_p variables V_i and W_i satisfying (9) and (11), and compute $F_i = V_i X_3^{-1}$ and $H_i = X_1^{-1} W_i$ $(i \in \mathbb{I}_{n_p})$.
(2) Assign the previous computed F_i and H_i in the n_p LMIs (22) and find X_1, $\{X_{2,i}, i \in \mathbb{I}_{n_p}\}$ and X_3.

Once $\{X_{2,i}, i \in \mathbb{I}_{n_p}\}$ are obtained, the n_p similarity transformations T_i can be computed and then the gain-scheduled controller is given by (12)-(15). This controller guarantees a performance level γ at any operating point, under the restriction that all local vertex controllers are recovered.

In terms of computational cost, the n_p LMIs (22) should be solved for variables $X_1 = X_1^T \in \mathbb{R}^{n\times n}$, $X_3 = X_3^T \in \mathbb{R}^{n\times n}$ and n_p variables $X_{2,i} = X_{2,i}^T \in \mathbb{R}^{n_q\times n_q}$. Previously the n_p variables (F_i, H_i) should be obtained from LMIs (9) and (11).

4 Example

A simple missile autopilot example is used to illustrate the procedure (Gahinet, Nemirovski, Laub & Chilali, 1995). The LPV plant has two states (six states when it is augmented with weights) and depends affinely on the parameter ρ ranging in $\mathcal{P} = [0.5,\, 4.0] \times [10,\, 106]$. Due to the affine dependency $\mathcal{P}$ was described by its four vertices $\mathcal{V}_1 = \{(0.5, 10), (4.0, 10), (0.5, 106), (4.0, 106)\}$. At each vertex, an LTI controller was designed using standard $\mathcal{H}_\infty$ tools. With these controllers and the system matrices of the plant at the four vertices, the gain matrices $\{(F_i, H_i)\, i \in \mathbb{I}_{n_p}\}$ were computed by solving the LMIs (9) and (11). The similarity transformations needed to construct the gain scheduled controller (5) were obtained from the LMIs (10) (Theorem 3.2) in the case of the stability preserving controller and from the LMIs (22) in the case of the controller designed for a performance level γ (Theorem 3.3). We have denoted each controller as $K_{\mathrm{qs}}(\rho)$ and $K_{\mathrm{perf}}(\rho)$, respectively.

In Figure 2, it can be observed the acceleration η when the closed loop system with several controllers is excited with a unitary step reference and the scheduling variable remain constant at the vertex ρ_1. The square marks correspond to local controller $K_1(s)$ and the solid line to the gain-scheduled controller $K_{\mathrm{perf}}(\rho_1)$. The coincident responses confirm that the interpolation scheme recovers the local controller at the vertices. In this figure, the response of an LPV controller computed with the procedure of Apkarian et al. (1995) can also observed (dashed line). In this particular example, only slight differences can be noted between the local designed controller and the LPV controller which is computed for the entire operating range. In general however, noticeable differences are expected.

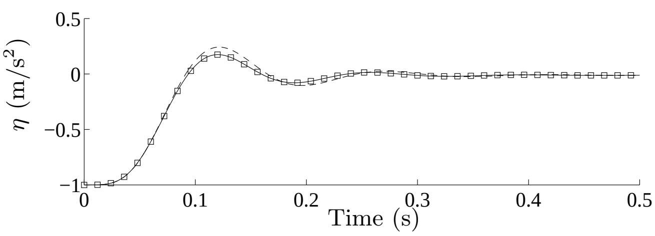

Figure 2. Comparison between the gain scheduled controller (solid line), an LPV controller (dashed line) and the local controller (square marks) at the vertex ρ_1

Figure 3 shows the response when the previous closed loop systems are excited with a step reference of amplitude 0.5 m/s^2 during a parameter trajectory depicted in Figure 3.b. The response of the closed loop system with $K_{\mathrm{qs}}(\rho)$ is indicated with dashed line and the response corresponding to $K_{\mathrm{perf}}(\rho)$ with solid line. With the aim of comparison, the response of the LPV controller (dashed and dotted line) is also included. The step occurs at $t = 0.5$ s when the parameter is at an intermediate point $\rho = (2.25, 10)$. The improvement in the performance achieved with the application of Theorem 3.3 with respect to the only stability preserving controller $K_{\mathrm{qs}}(\rho)$ becomes clear from observing this figure. The infinity norm of the closed loop system plus weights at the point $\rho = (2.25, 10)$ is 211 in case of $K_{\mathrm{qs}}(\rho)$, 2.18 in case of $K_{\mathrm{perf}}(\rho)$ and 0.46 in the case of LPV controller. As expected, the performance achieved by the LPV controller is better than $K_{\mathrm{perf}}(\rho)$. The LPV scheme aims to achieve a uniform performance in the entire operating envelope. In contrast, the proposed interpolation is intended for those cases where the system remains at the operating points most of the time and the transitions from one point to another rarely occur. On the other hand, notice that the option of simple linear interpolation without changing the realizations of the vertex controllers is unstable at $\rho = (2.25, 10)$.

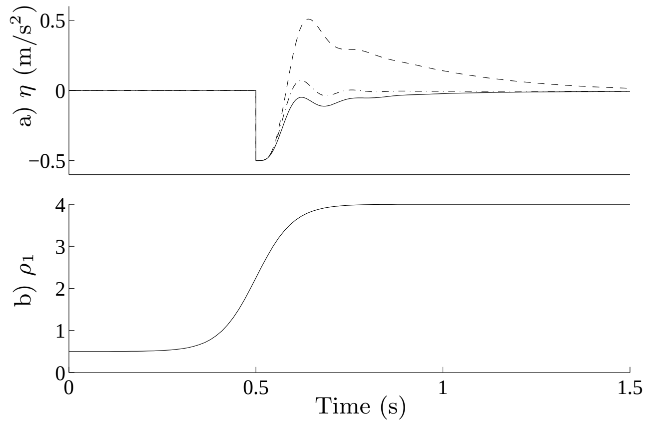

Figure 3. a) Step responses of the closed loop system with the controller K_{qs} (dashed line), with K_{perf} (solid line) and with LPV controller (dashed and dotted line) during a parameter trajectory $\rho_2 = 10$ and $\rho_1(t)$ in b)

5 Conclusions

A set of LMIs has been presented which modifies the realizations of a group of LTI designs in order to produce a gain-scheduled controller with quadratic stability and performance guarantees at intermediate interpolation points. The quadratic stability problem results in a convex optimization procedure and the performance guarantees require solving two consecutive convex problems using Youla parametrization arguments, in order to achieve the best performance in the intermediate points. A limitation on the global performance is the use of a block-diagonal Lyapunov function during the realizations computation.

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