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Multivariable PID control with set-point weighting via BMI optimisation ⋆

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Abstract

The paper focuses on the design of multivariable PID controllers with set-point weighting. The advantage of this PID structure is that the responses of the system to disturbances and to changes in the set-point can be adjusted separately. The proposed design methods rely on the transformation of the tuning of the controller gains into a static output feedback (SOF) problem. Hence, multivariable PID controllers can be designed by solving an optimisation problem with bilinear matrix inequalities (BMIs). The paper addresses the design of both time-invariant and gain-scheduled robust controllers. All of the tuning methods discussed through the paper are based on a PID structure with filtered derivative term, thus guaranteeing the well-posedness of the closed loop system.

Key words: PID control, Robust control, Gain scheduling, Bilinear matrix inequality (BMI), Linear parameter varying systems

1 Introduction

Although new and more powerful tools have been developed, PID control is still the most used control strategy in industrial applications. An attractive feature of PID controllers is their relatively simple and intuitive design. Moreover, the fixed structure of PID controllers has made possible the development of ready-made hardware modules and software packages for a quick and easy implementation (Li, Ang & Chong, 2006). For these reasons, PID controllers are commonly preferred even though more aggressive controllers can be obtained with other more sophisticated techniques.

The popularity of PID controllers has encouraged the formulation of a large number of methods for tuning the controller parameters (see *e.g.*, Astrom & Hagglund (2005); O'Dwyer (2006)). In MIMO plants with low interactions, it is possible to employ PID controllers in multiple SISO loops selected according to physical considerations or to a relative gain array (RGA) analysis. In this case, decentralised multivariable control, classical or indeed heuristic tuning methods can be used. However, in plants having strong interactions between the input and output pairs, it is necessary the use of centralised controllers designed with more sophisticated tuning methods usually based on optimisation notions (see *e.g.* Ruiz-Lopez, Rodriguez-Jimenes & Garcia-Alvarado (2006) and references therein). An approach, introduced in (Ge, Chiu & Wang, 2002; Zheng, Wang & Lee, 2002), consists in transforming the tuning of the controller parameters into a static output feedback (SOF) problem. Then, the controller parameters are determined by solving an optimisation problem with bilinear matrix inequalities (BMIs) for which several numerical algorithms are currently available (see *e.g.*, Goh, Safonov & Papavassilopoulus (1994); El Ghaoui, Oustry & AitRami (1997); Apkarian, Noll & Tuan (2003); Orsi, Helmke & Moore (2006)). A similar approach has also been used to formulate effective tuning methods for robust and gain-scheduled multivariable PID controllers (Mattei, 2001). It is interesting to note that these methods are actually particular cases of the procedures for designing optimal fixed-structure controllers. This is another motivation to examine these tuning methods.

Over the years, several modifications to the standard

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algorithm have been proposed with the aim of improving the performance and operativity of PID controllers. One of these alternative structures is the so-called PID with set-point weighting or two degree of freedom PID (2DOF-PID). The advantage of the 2DOF-PID structure is that the responses of the system to disturbances and to changes in the set-point can be adjusted separately. This characteristic results especially useful when the controller must accomplish several simultaneous specifications (Astrom & Hagglund, 2005). The 2DOF-PID structure is widely accepted for single-loop controllers. However, tuning methods for multivariable centralised 2DOF-PID controller are not commonly treated in the literature.

This paper introduces a tuning method for multivariable 2DOF-PID controllers based on BMI optimisation. Firstly, in Sec. 2, the design of 2DOF-PID controllers for linear time-invariant (LTI) plants is examined. Subsequently, Sec. 3 discusses the design of robust and gain-scheduled controller in the context of linear parameter varying (LPV) systems.

2 2DOF-PID control for LTI plants

Consider the following LTI plant

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_w w(t) + B_u u(t),\\ z(t) &= C_z x(t) + D_{zw} w(t) + D_{zu} u(t),\\ y(t) &= C_y x(t) + D_{yw} w(t),\end{aligned} \tag{1}$$

where $x \in \mathbb{R}^{n_s}$ is the state, $w \in \mathbb{R}^{n_w}$ is the disturbance, $u \in \mathbb{R}^{n_u}$ is the control input, $y \in \mathbb{R}^{n_y}$ is the measured output and $z \in \mathbb{R}^{n_z}$ is a fictitious signal used to assess the closed loop performance. It is assumed that the pairs (A, B_u) and (A, C_y) are stabilisable and detectable, respectively.

The purpose of this section is to provide a tuning method for a multivariable 2DOF-PID controller of the form

$$u(t) = K_p e_p(t) + K_i \int_0^t e(\eta)\,\mathrm{d}\eta + K_d \frac{\mathrm{d}e_d(t)}{\mathrm{d}t} \tag{2}$$

where $e_p(t) = K_b r(t) - y(t)$, $e(t) = r(t) - y(t)$,

$$\tau \dot{e}_d(t) = -e_d(t) + K_c r(t) - y(t), \tag{3}$$

$r(t) \in \mathbb{R}^{n_y}$ is the set-point or reference and K_p, K_i, K_d, K_b and K_c are constant matrices of dimension $n_u \times n_y$ to be found. The time constant τ, in the filtered derivative term (3), is fixed according to the required controller bandwidth.

The control action (2) can be expressed as

$$u(s) = G_{ff}(s)r(s) + G_c(s)y(s) \tag{4}$$

where

$$\begin{aligned}G_{ff}(s) &= K_{pr} + K_i \frac{1}{s} + K_{dr} \frac{s}{1+s\tau},\\ G_c(s) &= K_p + K_i \frac{1}{s} + K_d \frac{s}{1+s\tau},\end{aligned}$$

$K_{pr} = K_p K_b$ and $K_{dr} = K_d K_c$. Fig. 1 shows the closed loop system with the controller described by (4). It can be observed that the 2DOF-PID structure has different signal paths for the reference and for the controlled variable y. This feature permits to tune G_{ff} independently of G_c and then to improve the performance of the closed loop system. For example, the gains K_p, K_i and K_d can be tuned to accomplish stability and a good disturbance rejection whereas K_b and K_c to fulfil the specifications on the reference tracking.

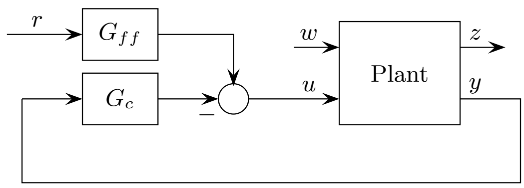

Fig. 1. Closed loop system with a 2DOF-PID controller

To transform the tuning of controller parameters into a SOF problem, we define the following transfer matrices

$$G_{c1} = \begin{bmatrix} 0 & -I_{n_y} \\ \frac{1}{s} I_{n_y} & -\frac{1}{s} I_{n_y} \\ 0 & -\frac{s}{1+s\tau} I_{n_y} \end{bmatrix}, \quad G_{c2} = \begin{bmatrix} I_{n_y} \\ \frac{s}{1+s\tau} I_{n_y} \end{bmatrix}$$

and reorganise the closed loop system as Fig. 2 depicts. Hence, the tuning of a 2DOF-PID controller reduces to find a SOF gain

$$K = \begin{bmatrix} K_{pr} & K_{dr} & K_p & K_i & K_d \end{bmatrix}. \tag{5}$$

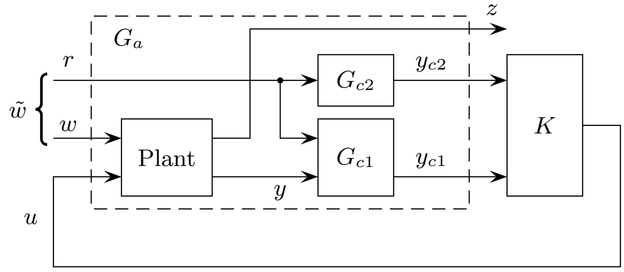

Fig. 2. Equivalent representation for the closed loop system in Fig. 1

In order to formulate a procedure to determine the constant gain K, let

$$\begin{aligned}\dot{x}_{c1}(t) &= A_{c1}x_{c1}(t) + B_{c11}r(t) + B_{c12}y(t),\\ y_{c1}(t) &= C_{c1}x_{c1}(t) + D_{c11}r(t) + D_{c12}y(t),\end{aligned} \tag{6}$$

and

$$\begin{aligned}\dot{x}_{c2}(t) &= A_{c2}x_{c2}(t) + B_{c2}r(t),\\ y_{c2}(t) &= C_{c2}x_{c2}(t) + D_{c2}r(t)\end{aligned} \tag{7}$$

be state-space realisations of the transfer matrices G_{c1} and G_{c2}, respectively. Therefore, the plant G_a comprised of G_{c1}, G_{c2} and the plant to be controlled (1) (see Fig. 2) is described by

$$\begin{aligned}\dot{x}_a(t) &= \tilde{A}x_a(t) + \tilde{B}_w\tilde{w}(t) + \tilde{B}_u u(t),\\ \tilde{z}(t) &= \tilde{C}_z x_a(t) + \tilde{D}_{zw}\tilde{w}(t) + \tilde{D}_{zu}u(t),\\ \tilde{y}(t) &= \tilde{C}_y x_a(t) + \tilde{D}_{yw}\tilde{w}(t),\end{aligned} \tag{8}$$

where $\tilde{y}^T = [y_{c2}^T \; y_{c1}^T]$, $\tilde{w}^T = [r^T \; w^T]$ and $\tilde{z}^T = [z^T \; z_l^T]$. The variable z_l includes some of the signals in Fig. 2 used as performance measurement. With these previous definitions the matrices of system (8) are

$$\begin{aligned}\tilde{A} &= \begin{bmatrix} A & 0 & 0\\ B_{c12}C_y & A_{c1} & 0\\ 0 & 0 & A_{c2}\end{bmatrix}, \; \tilde{B}_w = \begin{bmatrix} 0 & B_w\\ B_{c11} & B_{c12}D_{yw}\\ B_{c2} & 0\end{bmatrix},\\ \tilde{B}_u^T &= \begin{bmatrix} B_u^T \; 0 \; 0\end{bmatrix}, \quad \tilde{C}_z = \begin{bmatrix} C_z \; 0 \; 0\\ \hline C_{z_l}\end{bmatrix},\\ \tilde{D}_{zw} &= \begin{bmatrix} 0 \; D_{zw}\\ \hline D_{z_l w}\end{bmatrix}, \quad \tilde{D}_{zu} = \begin{bmatrix} D_{zu}\\ \hline D_{z_l u}\end{bmatrix},\\ \tilde{C}_y &= \begin{bmatrix} 0 & 0 & C_{c2}\\ D_{c12} & C_{c1} & 0\end{bmatrix}, \; \tilde{D}_{yw} = \begin{bmatrix} D_{c2} & 0\\ D_{c11} & D_{c12}D_{yw}\end{bmatrix}.\end{aligned} \tag{9}$$

Given the state-space realisation of the open loop system (8), we have to find the constant matrix K such that the closed loop system

$$\begin{aligned}\dot{x}_{cl}(t) &= (\tilde{A}+\tilde{B}_u K\tilde{C}_u)x_{cl}(t) + (\tilde{B}_z+\tilde{B}_u K\tilde{D}_{yw})\tilde{w}(t),\\ \tilde{z}(t) &= (\tilde{C}_z+\tilde{D}_{zu}K\tilde{C}_y)x_{cl}(t) + (\tilde{D}_{zw}+\tilde{D}_{zu}K\tilde{D}_{yw})\tilde{w}(t)\end{aligned}$$

is internally stable and fulfils the performance specifications. In the present discussion, the performance specifications are stated as a bound on the infinite norm of the transfer $\tilde{w}$ to $\tilde{z}$, *i.e.*, $\|T_{\tilde{z}\tilde{w}}(s)\|_\infty < \gamma$. Using the well-known Bounded Real Lemma, we can transform this specification in the frequency domain into a matrix inequality in the closed loop system matrices and in a symmetric positive definite matrix (Apkarian & Gahinet, 1995). Therefore, the application of this lemma to the closed loop system leads to the following theorem.

Theorem 1 *Given the plant (1), the 2DOF-PID controller (2) guarantees that the closed loop system is internally stable and $\|T_{\tilde{z}\tilde{w}}(s)\|_\infty < \gamma$ if there exist matrices K and $X = X^T > 0$ such that*

$$\begin{bmatrix} (\tilde{A}+\tilde{B}_u K\tilde{C}_y)^T X + (\star) & \star & \star\\ (\tilde{B}_w+\tilde{B}_u K\tilde{D}_{yw})^T X & -\gamma I & \star\\ \tilde{C}_z+\tilde{D}_{zu}K\tilde{C}_y & \tilde{D}_{zw}+\tilde{D}_{zu}K\tilde{D}_{yw} & -\gamma I\end{bmatrix} < 0 \tag{10}$$

where $\tilde{A}, \ldots, \tilde{D}_{yw}$ are given by (9). The symbol $\star$ denotes an element so that the previous matrix is symmetric.

The previous statement states that a set of 2DOF-PID controller gains can be found by solving an optimisation problem with a BMI constraint in K and X. This kind of optimisation problems is not convex and in general they prove to be more difficult to solve than the ones with LMI constraints. It must be noted that, except for very particular plants, any tuning procedure that aims to optimise the PID gains results nonconvex. This is an inevitable consequence of designing an optimal fixed-structure controller (Bhattacharyya, Chapellat & Keel, 1995). Nevertheless, in the literature can be found several number of algorithms for solving optimisation problems with BMIs in a reasonable time. Some of them search for local solutions such as (El Ghaoui et al., 1997; Orsi et al., 2006). Others permit to find the global solution at the expense of a larger computational effort (see *e.g.* Goh et al. (1994); Apkarian et al. (2003)). More recently, algorithms to obtain approximated solutions based on relaxations have been also proposed (see *e.g.* Henrion & Lasserre (2004)).

Observe that Theorem 1 also includes the design of particular cases such as 2DOF-PI and standard PID controllers. To this end, only the state-space realisations of G_{c1} and G_{c2} and the dimensions of the matrix K need to be modified. For tuning 2DOF-PI controllers, the last rows of the transfer matrices G_{c1} and G_{c2} must be eliminated and the constant gain matrix to be found is $K = [K_{pr} \; K_p \; K_i]$. This last version corresponds to a similar situation treated in (Zheng et al., 2002), with the difference that Theorem 1 considers a realisable version of the derivative term. The use of a realisable version for the derivative term in (2) guarantees that the closed loop system is well-posed and thus subsequent verifications or additional constraints are not necessary.

Remark 2 *Note that, in a similar way to $\mathcal{H}_\infty$ optimal control techniques, the performance signals can be passed through weighting functions to refine the performance specifications. So the different nature of the reference and disturbances can be considered in the tuning of the PID gains (Sanchez Peña & Sznaier, 1998). Moreover, the* *synthesis procedure can be further refined by including other performance criteria such as $\mathcal{H}_2$ norm, peak-to-peak norm, etc.*

Remark 3 *It must be noted that even though the Theorem 1 requires a model of the plant, it is not necessary an exact one. In fact, the modelling errors can be handled as usual in robust control. That is, the modelling errors can be covered with dynamic uncertainty and they can be taken into account in the design stage by selecting an input-output pair and a suitable weighting function (Sanchez Peña & Sznaier, 1998).*

Remark 4 *The computational effort increases with the complexity of the plant and the weighting functions since the number of unknowns increases with the dimension of the function X, a similar situation happens in full order $\mathcal{H}_\infty$ optimal control techniques. However, in spite of full order $\mathcal{H}_\infty$ controllers, in the case of the proposed tuning method the number of unknowns associated to the controller depends only on the dimensions of u and y. Therefore, in the case of PID controller the effort to compute online the control action is independent on the performance specifications. With respect to classical PID tuning procedures for multivariable plants, once the weighting functions are fixed, the proposed method permits to find the PID gains in only one step, without need of subsequent readjustments of the controller gains.*

3 2DOF-PID control for LPV plants

In this section, Theorem 1 is extended to linear parameter varying (LPV) plants. Commonly, the LPV plants arise after reformulating a nonlinear or time-varying system and they are closely related to gain scheduling techniques (Lu, Wu & Kim, 2005; Bianchi, De Battista & Mantz, 2006).

We will consider an LPV plant governed by

$$\begin{aligned}\dot{x}(t)&=A(\theta(t))x(t)+B_w(\theta(t))w(t)+B_u(\theta(t))u(t),\\ z(t)&=C_z(\theta(t))x(t)+D_{zw}(\theta(t))w(t)+D_{zu}(\theta(t))u(t),\\ y(t)&=C_y(\theta(t))x(t)+D_{yw}(\theta(t))w(t),\end{aligned} \tag{11}$$

The matrices $A(\cdot)$, $B_w(\cdot)$, $B_u(\cdot)$, $C_z(\cdot)$, $C_y(\cdot)$, $D_{zw}(\cdot)$, $D_{zu}(\cdot)$ and $D_{yw}(\cdot)$ are continuous functions of a vector of time varying parameters $\theta(t)$ taking values in a bounded set Θ. It is assumed that the pairs (A, B_u) and (A, C_y) are stabilisable and detectable, respectively, for all $\theta \in \Theta$. The transformation of the tuning of the 2DOF-PID controller parameters into a SOF problem, discussed in Sec. 2, is also valid in the case of LPV plants. Nevertheless, the method for computing the gain K must be modified.

In the case of an LPV plant, the stability and performance conditions must be satisfied for any possible trajectory $\theta(t)$. There are three possible cases to be analysed: robust, gain-scheduled and robust gain-scheduled controllers. The first is an LTI controller like the ones tuned in Sec. 2 but the matrix gain K must be determined to guarantee stability and to accomplish the performance specifications for any trajectory $\theta(t)$. The gain-scheduled controllers are actually an LPV system, *i.e.*, the gain K is a function of $\theta(t)$ which must be measured in real-time. Thus, a gain-scheduled controller can adjust itself according to the changes in the dynamics of the plant. Because of this feature, usually a gain-scheduled controller is less conservative than a robust one. Finally, in the most general situation, the robust gain-scheduled controllers, the parameter is assumed divided in $\theta^T = [\theta_m^T \ \theta_u^T]$ where θ_m is measured in real-time and the θ_u is uncertain. In this case, the matrix K varies only with the measured parameters θ_m. As a consequence, the controller can adapt itself to the changes in the plant dynamics due to θ_m and is robust in spite of the changes due to θ_u.

Now, the matrices corresponding to the state-space realisation of G_a (8) are matrix functions of the parameter θ. Therefore, the closed loop system is an LPV system and stability and performance must be assured for any possible trajectory $\theta(t)$. The stability of the closed loop system can be established using the concept of quadratic stability. The performance is assessed by means of the induced $\mathcal{L}_2$-norm of the operator $\tilde{w} \to \tilde{z}$,

$$\|T_{\tilde{z}\tilde{w}}\|_{i,2} = \sup_{\theta\in\mathcal{F}_\Theta} \sup_{\|w\|_2\neq 0} \frac{\|\tilde{z}\|_2}{\|\tilde{w}\|_2},$$

where $\mathcal{F}_\Theta$ is the set of possible trajectories $\theta(t)$ that take values in Θ. An extension of the Bounded Real Lemma is available to express the condition $\|T_{\tilde{z}\tilde{w}}\|_{i,2} < \gamma$ as a matrix inequality (Apkarian, Gahinet & Becker, 1995). The following theorem for tuning robust gain-scheduled 2DOF-PID controllers can be derived from applying this lemma to the closed loop LPV system.

Theorem 5 *Given the LPV plant (11), the 2DOF-PID controller (2), with K_p, K_i, K_d, K_b and K_c functions of θ_m, assures that the closed loop system is quadratically stable and $\|T_{\tilde{z}\tilde{w}}\|_{i,2} < \gamma$ if there exist a matrix function $K(\theta_m)$ and a matrix $X = X^T > 0$ such that, for any $\theta \in \Theta$,*

$$\begin{bmatrix} (\tilde{A}+\tilde{B}_u K \tilde{C}_y)^T X + (\star) & \star & \star \\ (\tilde{B}_w + \tilde{B}_u K \tilde{D}_{yw})^T X & -\gamma I & \star \\ \tilde{C}_z + \tilde{D}_{zu} K \tilde{C}_y & \tilde{D}_{zw} + \tilde{D}_{zu} K \tilde{D}_{yw} & -\gamma I \end{bmatrix} < 0, \tag{12}$$

where the functions $\tilde{A}(\cdot)$, $\tilde{B}_w(\cdot)$, $\tilde{B}_u(\cdot)$, $\tilde{C}_z(\cdot)$, $\tilde{C}_y(\cdot)$, $\tilde{D}_{zw}(\cdot)$, $\tilde{D}_{zu}(\cdot)$ and $\tilde{D}_{yw}(\cdot)$ are given by (9) and their dependence on θ have been dropped for brevity.

The optimisation problem is also nonconvex but in this case inequality (12) imposes an infinity number of constraints (one for each $\theta \in \Theta$). Nevertheless, when the matrices $\tilde{A}$, $\tilde{B}_w$, $\tilde{C}_z$ and $\tilde{D}_{zw}$ are affine functions of the parameter θ, $\tilde{B}_u$, $\tilde{C}_y$, $\tilde{D}_{zu}$ and $\tilde{D}_{yw}$ are constant matrices[4] and Θ is a convex polytope, to check the constraint (12) at the vertices of the polytope amounts to check it at all $\theta \in \Theta$ (Apkarian et al., 1995). This property reduces the previous procedure to an optimisation problem with a finite number of constraints. In this situation, Theorem 5 reduces to the following one.

Theorem 6 *Consider the LPV plant (11) with*

- $\tilde{A}(\cdot)$, $\tilde{B}_w(\cdot)$, $\tilde{C}_z(\cdot)$ *and* $\tilde{D}_{zw}(\cdot)$ *affine functions of the measured parameter* θ_m,
- $\tilde{B}_u$, $\tilde{C}_y$, $\tilde{D}_{zu}$ *and* $\tilde{D}_{yw}$ *constant matrices,*
- *and* Θ *a convex polytope of* k *vertices,* i.e., $\Theta = \mathrm{Co}\{\theta^{v1}, \ldots, \theta^{vk}\}$ *where* θ^{vj} *is the vertex* j.

Then the 2DOF-PID controller (2), with K_p, K_i, K_d, K_b *and* K_c *functions of the measured parameter* θ_m, *assures that the closed loop system is quadratically stable and* $\|T_{\tilde{z}\tilde{w}}\|_{i,2} < \gamma$ *if there exist a set of matrices* K^j *and a matrix* $X = X^T > 0$ *such that*

$$\begin{bmatrix} (\tilde{A}_j + \tilde{B}_u K_j \tilde{C}_y)^T X + (\star) & \star & \star \\ (\tilde{B}_{w,j} + \tilde{B}_u K_j \tilde{D}_{yw})^T X & -\gamma I & \star \\ \tilde{C}_{z,j} + \tilde{D}_{zu} K_j \tilde{C}_y & \tilde{D}_{zw,j} + \tilde{D}_{zu} K_j \tilde{D}_{yw} & -\gamma I \end{bmatrix} < 0$$

for all $j = 1, \ldots, k$, *where* $\tilde{A}_j = \tilde{A}(\theta^{vj})$, $\tilde{B}_{w,j} = \tilde{B}_w(\theta^{vj})$, $\tilde{C}_{z,j} = \tilde{C}_z(\theta^{vj})$ *and* $\tilde{D}_{zw,j} = \tilde{D}_{zw}(\theta^{vj})$.

The gains of the LPV controller are computed online from the K_j's and the measured θ_m according to the expression

$$K(\theta_m) = \sum_{j=1}^{k} \alpha_j K_j$$

where the α_j's must satisfy $\alpha_1 + \cdots + \alpha_k = 1$, $\alpha_j \geq 0$ and $\theta_m = \alpha_1 \theta_m^{v1} + \cdots + \alpha_j \theta_m^{vk}$.

LPV plants with more complex dependence on the parameter can be expressed as an affine system at expense of a more conservative design. Alternatively the number of constraints can be reduced to a finite number using gridding methods (Wu, Yang, Packard & Becker, 1996). This last option is less conservative, but it demands a larger computational effort and the controller results more complex.

[4] Note that the assumption that $\tilde{B}_u$, $\tilde{C}_y$, $\tilde{D}_{zu}$ and $\tilde{D}_{yw}$ are constant matrices is not restrictive since it can be always fulfilled by filtering the input $u(t)$ and/or the output $y(t)$ (Apkarian et al., 1995).

The previous theorems can be used to design other PID structures such as 2DOF-PI and standard PID. As mentioned in Sec. 2, the design of these cases just requires modifying the state-space realisations of G_{c1} and G_{c2}. Also, other performance criteria and the high frequency uncertainty can be handled as it was discussed in Remark 2 and 3, respectively.

4 Example

In order to illustrate the use of the proposed tuning methods, it is analysed the control of a quadruple-tank process (Johansson, 2000). The linearised dynamic equations of the plant are

$$\dot{x} = \begin{bmatrix} -\frac{1}{T_1} & 0 & \frac{A_3}{A_1 T_3} & 0 \\ 0 & -\frac{1}{T_2} & 0 & \frac{A_4}{A_2 T_4} \\ 0 & 0 & -\frac{1}{T_3} & 0 \\ 0 & 0 & 0 & -\frac{1}{T_4} \end{bmatrix} x + \begin{bmatrix} \frac{\gamma_1 k_1}{A_1} & 0 \\ 0 & \frac{\gamma_2 k_2}{A_2} \\ 0 & \frac{(1-\gamma_2)k_2}{A_3} \\ \frac{(1-\gamma_1)k_1}{A_1} & 0 \end{bmatrix} u,$$

$$y = \begin{bmatrix} k_c & 0 & 0 & 0 \\ 0 & k_c & 0 & 0 \end{bmatrix} x. \tag{13}$$

The state and the input are $x_j = h_j - h_j^0$ and $u_j = v_j - v_j^0$, respectively, where h_j is the height of tank j and v_j the voltage applied to the pump j, h_j^0 and v_j^0 denote the values at the operating point. The rest of the parameters are constant during the operation of the system. In the analysed situation, $A_1 = A_3 = 28\,\mathrm{cm}^2$, $A_2 = A_4 = 32\,\mathrm{cm}^2$, $a_1 = a_3 = 0.071\,\mathrm{cm}^2$, $a_2 = a_4 = 0.057\,\mathrm{cm}^2$, $k_c = 0.5\,\mathrm{V/cm}$, $g = 981\,\mathrm{cm/s}^2$, $k_1 = 3.33\,\mathrm{cm}^3/\mathrm{Vs}$, $k_2 = 3.35\,\mathrm{cm}^3/\mathrm{Vs}$, $\gamma_1 = 0.7$ and $\gamma_2 = 0.6$ (see (Johansson, 2000) for more details). The time constants are $T_j = (A_j/a_j)\sqrt{2h_j^0/g}$, $j = 1, \ldots, 4$.

First, a 2DOF-PI controller is tuned for the LTI plant corresponding to the operating point given by $v_1^0 = v_2^0 = 3$ V. For these pump voltages, $h_1^0 = 12.4$ cm, $h_2^0 = 12.7$ cm, $h_3^0 = 1.8$ cm and $h_4^0 = 1.4$ cm. The synthesis setup is sketched in Fig. 3. We have chosen the integral error e_i and the control input u as performance output z. With this performance output z, the tuning method will find in one step a set of controller gains that achieves a compromise between the regulation of y and the control effort. The signal $\tilde{w}$ consists of the reference r and a disturbance d entering at the input of plant. The weighting functions are $W_e = 0.4 I_2$, $W_{i_r} = \alpha I_2$, $W_{i_w} = (1-\alpha) I_2$ and

$$W_u(s) = \frac{0.2(s/10+1)}{s/100+1} I_2.$$

The parameter α allow us to place more o less emphasis on the reference tracking or on the disturbance rejection.

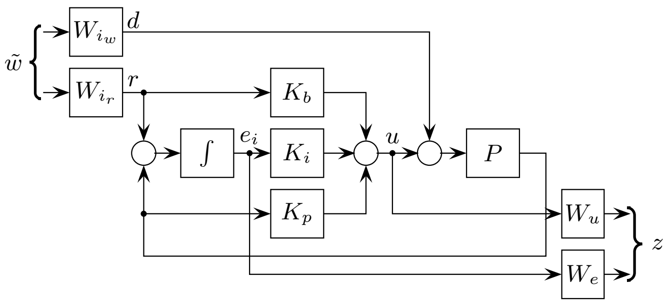

Fig. 3. Synthesis setup for tuning the 2DOF-PI in the quadruple-tank example

The SOF problem that arises in the PID tuning procedure was solved with the help of free-available algorithms Yalmip (Löfberg, 2004) and Lmirank (Orsi et al., 2006). Previously, using the elimination lemma (Apkarian & Gahinet, 1995) the BMI problem in Theorem 1 was expressed as an optimisation problem with a rank constraint. That is, the BMI problem in Theorem 1 is equivalent to find symmetric positive definite matrices X and Y such that

$$\mathcal{N}_x^T \begin{bmatrix} X\tilde{A} + (\star) & \star & \star \\ \tilde{B}_w^T X & -\gamma I & \star \\ \tilde{C}_z & \tilde{D}_{zu} & -\gamma I \end{bmatrix} \mathcal{N}_x < 0, \tag{14}$$

$$\mathcal{N}_y^T \begin{bmatrix} \tilde{A}Y + (\star) & \star & \star \\ \tilde{B}_w^T & -\gamma I & \star \\ \tilde{C}_z Y & \tilde{D}_{zu} & -\gamma I \end{bmatrix} \mathcal{N}_y < 0, \tag{15}$$

$$\mathcal{L} = \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0, \tag{16}$$

with the nonconvex constraint $\text{rank}(\mathcal{L}) = n$, where $\mathcal{N}_x = \ker([B_u \; 0 \; D_{zu}])$ and $\mathcal{N}_y = \ker([C_y \; D_{yw} \; 0])$. Since Lmirank does not support an objective function, the minimum γ was obtained by bisection. Once the matrix X is found, the gain K can be obtained by replacing X in (10) and solving the corresponding LMI. For the synthesis setup of Fig. 3, the algorithm obtains a gain K for a performance level $\gamma = 1.19$. The computations were performed with a PC with CPU Pentium IV 2.4 GHz and the cputime was 16.5 seconds.

Fig. 4 presents simulations corresponding to the nonlinear closed loop systems with the 2DOF-PI controller obtained with Theorem 1 (solid line). It is also shown the response of the closed loop system with a standard PI controller (Johansson, 2000). The system was excited with a step of 1 cm at each set-point followed by disturbances of 1 V. The top plot shows the height of tanks 1 and 2 and the bottom one the voltages applied to the pumps 1 and 2. It can be observed that the proposed 2DOF-PI achieves a faster response both to the change in the set-point and to the change in the disturbance.

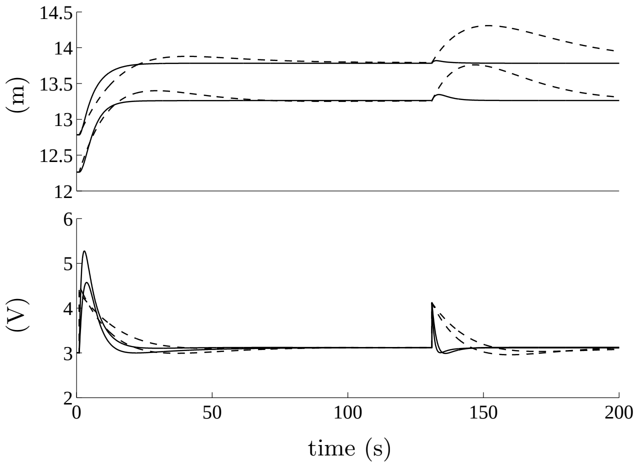

Fig. 4. Response of the closed loop system with an LTI 2DOF-PI controller (solid) and with a classical PI (dashed).

The best trade-off between the control objective can be attributed to the more flexible structure of the 2DOF-PI and the best performance to the optimisation procedure.

The methods discussed in Sec. 3 can be used to obtain robust or gain-scheduled controllers valid in certain operating range. The stationary values of the tank heights h_i^0 are functions of the voltages v_1^0 and v_2^0 (Johansson, 2000). Therefore, the time constants T_i and then the system matrices can be parameterised by the voltages v_1^0 and v_2^0. For brevity only the situation $v_1^0 = v_2^0 = v^0$ will be examined. So the time constants and the matrices of the model (13) are affine functions of a single parameter $\theta = 1/v^0$. The voltage v^0 was considered in the range of 1.6 to 3.6 V and hence the parameter θ takes values in the polytope $\Theta = \text{Co}\{0.278, 0.625\}$.

Fig. 5 presents the responses of the nonlinear closed loop system with three controllers. The solid line corresponds to a 2DOF-PI controller and the dashed line to a 1DOF-PI, both designed with the proposed method. On the other hand, the dotted line corresponds to a full order controller synthesised according to (Apkarian et al., 1995). The synthesis setup used was the same of the LTI case (see Fig. 3). The SOF problem was solved with the previous algorithm but with (14)-(16) checked at the vertices of Θ. The performance level γ was 1.66 and the cputime was 11.1 seconds. In Fig. 5 it can be observed again that, with similar disturbance rejection, the 2DOF controllers achieve a faster response to changes in the set-point. Also, it can be seen that the full order controller does not provide an appreciable improvement with respect to the 2DOF-PI controller. This is a case where a 2DOF-PI tuned with the proposed method permits a simpler implementation without sacrifices performance.

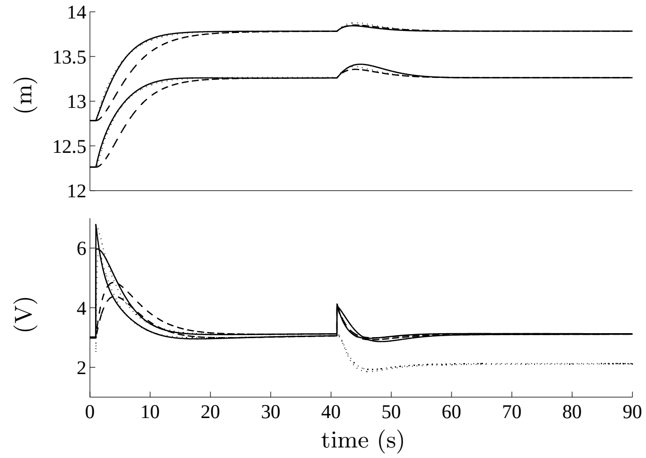

Fig. 5. Response of the closed loop system with three gain-scheduled controllers. 2DOF-PI (solid), 1DOF-PI (dashed) and full order controller (dotted)

5 Conclusions

In this paper, tuning methods for centralised multivariable 2DOF-PID controllers have been proposed. Firstly, the design of LTI controllers was discussed and then the tuning of 2DOF-PID controller for LPV plants. In both cases, the proposed methods include the design of other PID structures such as the 2DOF-PI and the standard PI(D) control. In the case of LPV plants, the proposed methods allow to tune both time invariant and robust gain-scheduled controllers according to the possibility to measure the time varying parameters of the plant. In all the methods presented in the paper, it is possible to consider high frequency uncertainty so that the tuning of the controllers can be based on approximated models of the plant. Furthermore, the formulation in the context of BMI optimisation enables to extend to other criteria even multi-objective.

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